ON SPACE OF OPEN QUOTIENT OBJECTS OF THE CANTOR SET

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Let \( f_1 : X \rightarrow Z_1 \) and \( f_2 : X \rightarrow Z_2 \) be continuous onto maps of compact Hausdorff spaces. We say that \( f_1 \) is equivalent to \( f_2 \) (notation \( f_1 \sim f_2 \)), if there exists a homeomorphism \( h : Z_1 \rightarrow Z_2 \) such that \( f_2 = h \circ f_1 \).

Clearly, \( \sim \) is an equivalence relation on the class of all continuous onto maps. The class of \( \sim \) that contains \( f \) is denoted \([f]\) and is called the quotient object of the space \( X \).

The set of all quotient objects of the space \( X \) is denoted by \( \Phi(X) = \{[f] \mid f : X \rightarrow Z\} \).

The set \( \Phi(X) \) was first introduced by E. Shchepin in his article *Functors and countable powers of compact* [1]. It plays a significant role in his investigations of nonmetrizable compact Hausdorff spaces by methods of inverse spectral. By \( \Psi(X) \) we denote the subset in \( \Phi(X) \) generated by the open maps.

Let \( f : X \rightarrow Z \) be an open continuous onto map of compact Hausdorff spaces.

Recall that a map \( f : X \rightarrow Z \) is open if the image of every open set in \( X \) is open in \( Z \).

It is known that the openness of \( f : X \rightarrow Z \) is equivalent to the continuity of \( f^{-1} : Z \rightarrow \exp X \), where \( \exp X \) denotes the hyperspace (the set of all nonempty closed subsets of \( X \)) endowed with the Vietoris topology.

A base of this topology is formed by the sets:

\[
\langle U_1, U_2, \ldots, U_n \rangle = \{ A \in \exp X \mid A \subset \bigcup_{i=1}^{n} U_i \text{ and } A \cap U_i \neq \emptyset \text{ for all } i \in \{1, 2, \ldots, n\} \}
\]

where the sets \( U_1, U_2, \ldots, U_n \) run through the topology of \( X \) and \( n \in \mathbb{N} \).

Consider the set

\[
\tilde{\Psi}(C) = \{ (f) \mid f : C \rightarrow f(C) \text{ is an open map } \}
\]

of the open quotient objects of the Cantor set \( C \). We identify every class of equivalent maps \( (f) \in \Psi(C) \) with the family \( \{ f^{-1}(y) \mid y \in f(C) \} = \{ f^{-1}(f(x)) \mid x \in C \} \) of closed subsets of \( \exp C \). Since \( f \) is open, we show that \( (f) \in \exp^2 X \).

The constructed correspondence is one to one and continuous which allows us to embed the set \( \Psi(C) \) into the space \( \exp^2(C) \) endowed with the Vietoris topology.

Then a base neighborhood of element \( (f) \in \Psi(C) \) is the set

\[
O(f) = \langle \langle U_{11}, U_{12}, \ldots, U_{1n_1} \rangle, \ldots, \langle U_{k1}, U_{k2}, \ldots, U_{kn_k} \rangle \rangle,
\]

where \( U_{ij} \) are open subsets of the space \( C \) such that:

1) for all \( x \in C \) there exists \( i \in \{1, 2, \ldots, k\} \), such that \( f^{-1}(f(x)) \in \langle U_{11}, U_{12}, \ldots, U_{1n_1} \rangle \);
2) for all \( i \in \{1, 2, \ldots, k\} \) there exists \( x \in C \) such that \( f^{-1}(f(x)) \in \langle U_{i1}, U_{i2}, \ldots, U_{in_i} \rangle \).

The obtained topological space of open quotient maps of the space \( C \) is denoted by \( \Psi(C) \).

Therefore, \( \Psi(C) \) is a subset in the space \( (\exp^2 C, \tau_V) \). It is known that \( (\exp^2 C, \tau_V) \) is a compact, zero dimensional, perfect, completely metrizable space.

We prove that
Proposition 1. The set $\Psi(C)$ of open quotient objects of the Cantor set is a dense subset in the $(\exp^2 C, \tau_V)$.

Proposition 2. The space $\Psi(C)$ is G$_\delta$ subset of the $(\exp^2 C, \tau_V)$.

Proposition 3. The space $\Psi(C)$ is a nowhere locally compact subset of the $(\exp^2 C, \tau_V)$.

Theorem 1. The space of open quotient objects of the Cantor set is homeomorphic to the space of irrationals.

References


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