Spaces with “few” norm attaining functionals and no proximinal subspaces of codimension 2

Vladimir Kadets
Joint results with G. López, M. Martín and D. Werner

V.N.Karazin Kharkiv National University, Ukraine

IDAT 2019, Ivano-Frankivsk
Norm-attaining functionals

Let $X$ be a real Banach space, $X^*$ be its dual space. Recall that a functional $x^* \in X^*$ is **norm-attaining** ($x^* \in NA(X)$ for short) if there is an $x \in S_X$ with

$$x^*(x) = \|x^*\|.$$

In reflexive spaces $NA(X) = X^*$, in non-reflexive ones there are functionals that do not attain their norm. Nevertheless, $NA(X)$ cannot be too small: the famous Bishop-Phelps theorem (1961) says that for every Banach space $X$ the corresponding set $NA(X)$ is dense in $X^*$.

A good example that illustrates these effects is the space $c_0$, for which $NA(c_0)$ is the set $c_{00}$ of those $y = (y_1, y_2, \ldots) \in \ell_1$ that have finitely many non-zero coordinates.
A subset \( Y \) of a Banach space \( X \) is said to be \textit{proximinal}, if for every \( x \in X \) there is a \( y \in Y \) such that

\[
\| x - y \| = \text{dist}(x, Y).
\]

Every closed subspace of codimension 1 in a Banach space \( X \) is of the form

\[
\ker x^* = \{ x \in X : x^*(x) = 0 \},
\]

where \( x^* \in S_{X^*} \). The proximinality of \( \ker x^* \) is equivalent to the condition that \( x^* \in \text{NA}(X) \). So, the Bishop-Phelps theorem implies that in every Banach space there are “many” proximinal subspaces of codimension 1.
Norm-attaining operators

When instead of functionals one considers operators, the situation changes. Although between any two Banach spaces $X, Y$ there are norm-attaining non-zero operators (say, rank one operators $x^* \otimes y$ with $x^* \in \text{NA}(X)$), the set $\text{NA}(X, Y)$ of norm-attaining operators is not always dense in the space $L(X, Y)$ of all linear operators acting from $X$ to $Y$. 
Three important questions

(i) Is it true that for every infinite-dimensional Banach space $X$ there is a norm-attaining surjective operator acting from $X$ onto two-dimensional Hilbert space?

(ii) Does every infinite-dimensional Banach space $X$ contain a proximinal subspace of codimension 2? (Ivan Zinger, 1974)

(iii) Is it true that for every infinite-dimensional Banach space $X$ the dual space contains a two-dimensional linear subspace consisting of norm-attaining functionals? (Gilles Godefroy, 2001)

The positive answer to (ii) would imply the positive answer to (iii) which in its turn would imply the positive answer to (i).
Charles J. Read in his brilliant paper “Banach spaces with no proximinal subspaces of codimension 2” (2018, preprint version – 2013) constructed an equivalent norm $||| \cdot |||$ on $c_0$ such that the space $\mathcal{R} = (c_0, ||| \cdot |||)$ does not have proximinal subspaces of codimension 2, thus answering in negative the problem (ii).

Martin Rmoutil (2017) demonstrated that the same space $\mathcal{R}$ gives the negative solution to the problem (iii).

The most challenging problem (i) remains open.
After these results one more natural problem arises:

(iv) What Banach spaces $X$ possess an equivalent \textit{Read norm} that is, such a norm $p$, that $NA(X, p)$ contains no two-dimensional linear subspaces (and consequently $(X, p)$ has no proximinal subspaces of codimension 2)?

In this talk I am going to explain our recent joint results with Ginés Lopez, Miguel Martín and Dirk Werner.

Developing Read’s ideas by adding a new technical tool of “modest subspaces”, we present a (hopefully) understandable construction of equivalent norms that enables to make some steps toward the above problem (iv).
References


Operator ranges

A linear subspace $Y$ of a Banach space $E$ is said to be an operator range if there is an infinite-dimensional Banach space $M$ and a bounded injective operator $T : M \to X$ such that $T(M) = Y$.

Operator ranges in Banach spaces attracted attention of many mathematicians because the domain of a closed operator between Banach spaces is an operator range and every operator range is the domain of some closed linear operator.

Remark that $Y$ is an operator range if and only if there is a complete norm on $Y$ which is stronger than the restriction of the given norm of $X$ to $Y$. A non-closed operator range is not barreled; finally, the injectivity of $T$ in the definition of operator range can be substituted by the condition $\dim Y = \infty$, because for every non-injective $T : M \to X$ there is an injective $\tilde{T} : M/\ker T \to X$ with the same range.
Lemma. Let $Y \subset X$ be a separable operator range. Then there exists an injective norm-one linear operator $T : \ell_1 \rightarrow Y$ such that the set $\left\{ \frac{T e_n}{\|T e_n\|} : n \in \mathbb{N} \right\}$ is dense in $S_Y$. 
Modest subspaces

A linear subspace \( Z \subset X \) is said to be \textit{modest} if there is a separable dense operator range \( Y \subset X \) such that \( Z \cap Y = \{0\} \). If \( X = E^* \) is a dual space, a linear subspace \( Z \subset X \) is said to be \textit{weak-star modest} if there is a separable weak-star dense operator range \( Y \subset X \) such that \( Z \cap Y = \{0\} \).

Observe that in the definition of (weak-star) modest subspace \( Z \subset X \), the space \( M \) which is the domain of \( T: M \rightarrow Y \) can be supposed to be separable: just substitute \( M \) by the closed linear span in \( M \) of the preimage of a countable dense subset of \( Y \).
Modesty of $c_{00}$ in $\ell_1$

**Lemma.** There is a dense operator range $Y \subset \ell_1$ such that every non-zero $y \in Y$ has a finite number of zero coordinates.

**Proof.** Let $A(\mathbb{D})$ be the disc algebra, viewed as a real Banach space. Denote $A_r(\mathbb{D}) \subset A(\mathbb{D})$ the closed real subspace consisting of those $f$ that take real values on the real axis, and take $t_n = 2^{-n}$, $n \in \mathbb{N}$. We define $T : A_r(\mathbb{D}) \rightarrow \ell_1$ by

$$Tf = \left( f(t_1), \frac{1}{2} f(t_2), \frac{1}{4} f(t_3), \ldots \right).$$

Then, the identity theorem for analytic functions implies that in $Y := T(A_r(\mathbb{D}))$ every non-zero element has a finite number of zero coordinates (if any). It remains to demonstrate the density of $Y$ in $\ell_1$. 
Density of $Y$ in $\ell_1$

It is sufficient to show that every element $e_m$ of the canonical basis of $\ell_1$ belongs to the closure of $Y$.

Indeed, for a fixed $m \in \mathbb{N}$, consider the function

$$f(z) = 1 - (z - t_m)^2.$$ 

This $f \in A_r(\mathbb{D})$ takes the value 1 at $t_m$ and $0 < f(t_k) < 1$ for all $k \neq m$. Denote $f_n = f^n \in A_r(\mathbb{D})$. Then, $\lim_{n \to \infty} f_n(t_m) = 1$, and $\lim_{n \to \infty} f_n(t_k) = 0$ for $k \neq m$, hence by Lebesgue dominated convergence theorem $\lim_{n \to \infty} Tf_n = e_m/2^{m-1}$, and $e_m$ is in the closure of $Y$. 
The main result

**Theorem.** Let $X$ be a Banach space such that $\text{lin}(\text{NA}(X))$ is a weak-star modest subspace of $X^*$. Then $X$ possesses an equivalent Read norm $p$. Moreover, $p$ can be chosen in such a way that, given two linearly independent functionals $x^*, z^* \in \text{NA}(X, p)$ with $p^*(x^*) = p^*(z^*) = 1$, one has $x^* + z^* \notin \text{NA}(X, p)$ or $x^* - z^* \notin \text{NA}(X, p)$.

**Proof.** Let $Y \subset X^*$ be a separable weak-star dense operator range with $\text{lin}(\text{NA}(X)) \cap Y = \{0\}$. We may assume that $Y = T(\ell_1)$, where $T: \ell_1 \rightarrow X^*$ is an injective bounded linear operator such that the set $\left\{ \frac{T_e}{\|T_e\|} : n \in \mathbb{N} \right\}$ is dense in $S_Y$. 
The definition of norm $p$

Take a sequence $\{r_n\}$ of positive reals such that $\sum_{k\in\mathbb{N}} r_k < \infty$, and denote $v^*_n = T(e_n)$. The desired equivalent norm $p$ on $X$ will be

$$p(x) = \|x\| + \sum_{n\in\mathbb{N}} |r_n v^*_n(x)|.$$ 

The unit ball of the dual space may be described as follows:

$$B(X,p)^* = B_X^* + \sum_{n\in\mathbb{N}} r_n [-v^*_n, v^*_n].$$

In other words, every $x^* \in B(X,p)^*$ is of the form

$$x^* = x_0^* + \sum_{n\in\mathbb{N}} t_n r_n v^*_n$$

with $x_0^* \in S_X^*$ and $t_k \in [-1, 1]$. 
Norm-attaining functionals on $(X, p)$

Consider two linearly independent functionals $x^*, z^* \in NA(X, p)$ with $p^*(x^*) = p^*(z^*) = 1$, and let $x, z \in X$ with $p(x) = p(z) = 1$ such that $x^*(x) = z^*(z) = 1$. There are representations

$$x^* = x_0^* + \sum_{n \in \mathbb{N}} t_n r_n v_n^*, \quad z^* = z_0^* + \sum_{n \in \mathbb{N}} \tau_n r_n v_n^*. \quad (1)$$

with $t_k, \tau_k \in [-1, 1]$ such that

1. $x_0^*, z_0^* \in S_{X^*} \cap NA(X)$,

2. for every $n \in \mathbb{N}$ where $v_n^*(x) \neq 0$ one has $t_n = \text{sign} \; v_n^*(x)$, and for every $n \in \mathbb{N}$ where $v_n^*(z) \neq 0$ one has $\tau_n = \text{sign} \; v_n^*(z)$. 
The selection of sign

Let $\theta = \pm 1$ be a sign such that $x \neq \theta z$. First, remark that, by weak-star density of $Y$, the set of restrictions of functionals from $Y$ to the linear span of $x$ and $z$ is the whole $(\text{lin}\{x, z\})^*$. So, there is $y_0^* \in S_Y$ such that $y_0^*(x) < 0$ and $y_0^*(\theta z) > 0$. Consequently, there is a neighbourhood $U_0$ of $y_0^*$ in $S_Y$ such that for all $y^* \in U_0$, we have $y^*(x) < 0$ and $y^*(\theta z) > 0$. Then, for all those $n \in \mathbb{N}$ for which $\frac{v_n^*}{\|v_n^*\|} \in U_0$, we have that

$$t_n + \theta \tau_n = \text{sign } v_n^*(x) + \theta \text{ sign } v_n^*(z) = 0.$$
How \( x^* + \theta z^* \) could attain its norm?

We are going to demonstrate that \( x^* + \theta z^* \notin \text{NA}(X, p) \). Assume to the contrary that there is \( e \in X \) with \( p(e) = 1 \) at which \( x^* + \theta z^* \) attains its norm, that is \( (x^* + \theta z^*)(e) = p^*(x^* + \theta z^*) \). Again, one can write

\[
\frac{x^* + \theta z^*}{p^*(x^* + \theta z^*)} = h_0^* + \sum_{n \in \mathbb{N}} s_n r_n v_n^*,
\]

with \( s_k \in [-1, 1] \), \( h_0^* \in \text{NA}(X) \), and for every \( n \in \mathbb{N} \) where \( v_n^*(e) \neq 0 \), one has \( s_n = \text{sign } v_n^*(e) \).
The behavior of coefficients

Since $Y$ is weak-star dense, it cannot be contained in a weak-star closed hyperplane. Consequently, the set $S_Y \cap \{ h^* \in X^* : h^*(e) = 0 \} = S_Y \cap \{ h^* \in Y : h^*(e) = 0 \}$ is nowhere dense in $S_Y$. This implies that there is a non-empty relatively open subset $U_1 \subset U_0$ of $S_Y$ which does not intersect the hyperplane $\{ h^* \in Y : h^*(e) = 0 \}$. Denote

$$N_1 = \left\{ n \in \mathbb{N} : \frac{v_n^*}{\|v_n^*\|} \in U_1 \right\},$$

which is non-empty by density of $\{ \frac{v_n^*}{\|v_n^*\|} : n \in \mathbb{N} \}$ in $S_Y$. Then, for every $n \in N_1$ the conditions

$$t_n + \theta \tau_n = 0, \quad s_n = \text{sign } v_n^*(e) \neq 0 \quad (3)$$

hold true at the same time.
Combining the representations

Now, from equations (1) and (2) we get

\[
0 = x^* + \theta z^* - p^*(x^* + \theta z^*) \frac{x^* + \theta z^*}{p^*(x^* + \theta z^*)} \\
= (x_0^* + \theta z_0^* - p^*(x^* + \theta z^*) h_0^*) \\
+ \sum_{n \in \mathbb{N}} (t_n + \theta \tau_n - p^*(x^* + \theta z^*) s_n) r_n v_n^*.
\]

In other words,

\[
x_0^* + \theta z_0^* - p^*(x^* + \theta z^*) h_0^* \\
= - T \left( \sum_{n \in \mathbb{N}} (t_n + \theta \tau_n - p^*(x^* + \theta z^*) s_n) r_n e_n \right).
\]
\[ x_0^* + \theta z_0^* - p^*(x^* + \theta z^*)h_0^* = -T \left( \sum_{n \in \mathbb{N}} (t_n + \theta \tau_n - p^*(x^* + \theta z^*)s_n) r_n e_n \right). \]

The left hand side belongs to \( \text{lin}(\text{NA}(X)) \), the right hand side belongs to \( Y \), so both of them are equal to zero. Since \( T \) is injective, and \( \{ e_n \}_{n \in \mathbb{N}} \) form a basis of \( \ell_1 \), this means that all \( t_n + \theta \tau_n - p^*(x^* + \theta z^*)s_n \) are equal to zero. On the other hand, as we remarked before, for every \( n \in N_1 \) we have \( t_n + \theta \tau_n = 0 \) and \( s_n \neq 0 \), so \( t_n + \theta \tau_n - p^*(x^* + \theta z^*)s_n \neq 0 \). This contradiction completes the proof.
More open questions

**Corollary.** Let $X$ be a Banach space containing an isomorphic copy of $c_0$ and possessing a countable norming system of functionals. Then $X$ admits an equivalent Read norm, i.e. a norm for which the corresponding set of norm attaining functionals contains no linear subspaces of dimension two.

**Problem 1.** Does every non-reflexive separable Banach space admit an equivalent Read norm? In particular, does $\ell_1$ admit an equivalent Read norm?

All separable infinite-dimensional $C(K)$ spaces, as well as $\ell_\infty$ satisfy the conditions of the above Corollary, so they admit an equivalent Read norm. On the other hand, it can be demonstrated that $\ell_\infty(\Gamma)$ for uncountable $\Gamma$ does not have any equivalent Read norm. This leads to the following open question:

**Problem 2.** For what compact spaces $K$ the corresponding $C(K)$ possesses an equivalent Read norm?
Thank you for your attention!